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ABSTRACT

Independent neighborhood schools in inner-city areas serve primarily minority students. They are in a position to assist American educators in understanding the best methods of teaching minorities who usually do not reach their full academic potential in public schools. Teachers in independent schools use culture and sometimes religion as a basis for motivational and management techniques. The students use their own indigenous images to shape their intellectual, emotional and creative development. In 1984 a training seminar using this cultural model was developed to provide these teachers with a better understanding and more strategies for teaching mathematics. This second volume of a two-volume series bridges the gap between theoretical perspectives and mathematic principles by presenting specific problems, concepts, vocabulary, activities and historical information concerning mathematics. The strategies are arranged in the following sections: (1) A Training Course for Mathematics Teachers in Elementary Grades (Gerald Chachere and Tepper Gill); (2) Teaching the Language of Mathematics (Mu'minah M. Saleem); and (3) Early African Mathematics for the Classroom (Walter M. Young). (VM)

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Teaching Mathematics

Volume II

Strategies in Mathematics

Tepper Gill and Gerald Chachere
Mu'minah Saleem • Walter Young
Edited by
Oswald M.T. Ratteray

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ERRATA

Page 13: QUOTIENTS

The lower part of the first long division, beginning with the number "156", is not properly aligned. The "6" should be in the ones' place, the "5" in the tens' place, and so on, as follows:

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  313
12)3756
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  120
   36
   36
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Page 15, line 13:

The correct spelling is "pharaohs."

A Training Course For Mathematics Teachers in Elementary Grades

Gerald Chachere, Ph.D., and Tepper Gill, Ph.D.

Students often approach problem solving in mathematics with a sense of fear, rather than exploration. They can, however, overcome this fear and develop competence in mathematics if their teachers understand some of the conceptual, technical, and process issues that serve as barriers to learning.

The following material is not presented as a self-study manual. It is a guide for instructors who are offering teachers in elementary grades a deeper understanding of important mathematical rules and principles. It shows how exploring these rules and principles can lead teachers to discover many relationships between numbers, beginning with number theory, then expanding the discussion to number systems, probability and statistics, and geometry.

Number Theory

Number theory expresses the basic relationships between counting numbers.

Divisibility Factors and Multiples

DEFINITION 1: A counting number n is divisible by a counting number s "if and only if" there is a counting number k, such that:

$$n = s \times k.$$

DEFINITION 2: If the counting number n is divisible by s, then we call s a **factor** of n, and we call n a **multiple** of s. Furthermore, a counting number is said to be divisible by another counting number if, on division, the remainder is zero.

OBJECTIVES: To learn how to factor a given number, how to determine if a given number is or is not a multiple of another number, and how to determine if a number is or is not divisible by another number.

Problems

NOTE: The first step in solving a problem is to restate the definition. In Problem A, for example, we must find k, and we can restate the problem as $24 = 8 \times k$. In Problem B, we must find all pairs s and k, so that $24 = s \times k$.

Problems for Definition 1:

A. Is the number 24 divisible by 8?

B: What are all the numbers by which 24 is divisible?

Problems for Definition 2:

A. Discover some general properties of all numbers divisible by 2, 4, 3, 5, 6, 7 or 11.

B. Study the multiplication table for 9 and discover some of its properties.

C. Study the divisibility of the sums and products of consecutive counting numbers.

Prime Numbers and Prime Factorization

A **prime number** is a counting number that has only two factors. Any number greater than 1 that is not prime is called a **composite number**. (NOTE: The number 1 is neither prime nor composite.)

OBJECTIVES: To know the definition of prime numbers, to learn how to classify a number as a prime or composite number, and to find the prime factors of any number.

Problems

A. Find the factors of 1547 and the factors of 13. Are they prime?

B. Study $n^2 - n + 41$ for $n = 1, 2...$ up to 40, and discuss their relationship to prime numbers.

C. A fundamental theorem of arithmetic is that every composite number may be expressed uniquely as a product of prime factors, if the order of the factors is disregarded. Discuss this statement and construct: 1) the sieve of Erathosthenes; and 2) factor trees.

D. Show that the number of primes is infinite and discuss Fermat numbers.

Greatest Common Factor and Least Common Multiple

DEFINITION 1: A **common factor** of two numbers n and m is any number k which is a factor of both. The **greatest common factor** (GCF) is the largest number k which is a factor of both n and m. The GCF is sometimes called the **greatest common divisor** (GCD).

DEFINITION 2: A **least common multiple** (LCM) of two numbers n and m is the smallest number that is a multiple of both m and n.

The LCM is also known as the **least common denominator** (LCD), when it simplifies the addition or subtraction of two or more fractions.)

OBJECTIVES: 1. To find the greatest common factor and use the concept of GCF to reduce a fraction to its lowest terms; and 2. to find the least common multiple and use the LCM to simplify the addition of fractions.

Problems

Problems for Definition 1:

A. Find the GCF of 60 and 5280; 93 and 155.

B. Reduce to their lowest terms -60/168; 112/480.

C. Show that there is a systematic process for reducing fractions and for simplifying them.

Problems for Definition 2:

A. Find the LCM of 6 and 9; 3850 and 5280; 12, 18 and 20.

B. Simplify the following:

$$\frac{37}{5280} - \frac{18}{3850}\ ;\ \frac{1}{3} - \frac{1}{2}\ ;\ \frac{1}{4} + \frac{1}{6}\ ;$$

$$\frac{2}{3} + \left(\frac{3}{4} + \frac{7}{8}\right)$$

C. Also study deficient numbers, perfect numbers, and abundant numbers.

Number Systems

A number system consists of a set of numbers and one or more binary operations.

The binary operations are addition (+), subtraction (-), multiplication (x), and division (÷). They are used to combine two elements of a set so as to form a third element of the same set.

A number system can be *closed*, or it can have *associative*, *commutative*, *identity*, *inverse*, or *distributive* properties.

The following list contains possible properties of a number system, with S the set of numbers and (*) or (#) the binary operations on S.

1. $(S,*)$ is **closed** if $a*b$ is in S, when a and b are in S.

2. $(S,*)$ is **associative** if $(a*b)*c = a*(b*c)$, when a, b, and c are in S.

3. (S,*) is **commutative** if $a*b = b*a$, when a and b are in S.

4. $(S,*)$ has an **identity** e, if e is in S, and $a*e = e*a = a$, whenever a is in S.

5. Assuming $(S,*)$ has the identity property, b in S is the **(*)-inverse** of a in S, if $a*b = b*a$ = the identity element of $(S,*)$.

6. In $(S,*,\#)$, * is **distributive** over # in S if $a*(b\#c) = (a*b)\#(a*c)$, when a, b, and c are in S.

There are also more general mathematical systems, i.e., systems in which S is a family of subsets.

Properties of Whole and Counting Numbers

Whole numbers are the numbers 0, 1, 2, 3, etc., represented by the letter W. **Counting numbers**, also called *natural numbers*, are 0, 1, 2, 3, etc., and can be designated by N.

OBJECTIVE: To demonstrate which properties hold, do not hold, or do not apply in N and W, utilizing binary operations +, x, -, and ÷.

Problems

A. Which properties hold, do not hold, or do not apply to the following number systems:

$(N,+)$ (N,x) $(N,-)$ and $(N,\div)$
$(W,+)$ $(W,-)$ $(W,-)$ and $(W,\div)$

B. For which number systems, involving only N and W, does the distributive property hold?

Properties of Integers

Similarly, assume that Z represents a set of integers (e.g. ..., -3, -2, -1, 0, 1, 2, 3,...)

Problems

A. Which properties hold, do not hold, and do not apply to: $(Z,+)$ (Z,x) $(Z,-)$ and $(Z,\div)$?

B. Which elements of Z have (*)-inverses where * can be +, x, -, or ÷?

For which number systems does it make sense to ask such a question?

Properties of Rational Numbers

Similarly, assume that Q represents a set of rational numbers or fractions, i.e., the set of numbers of the form a/b, where a and b are integers but b = zero.

Problems

A. Which properties hold, do not hold, and do not apply to: $(Q,+)$ (Q,x) $(Q,-)$ and $(Q,\div)$?

B. Which elements of Q have $(*)$-inverses where $*$ can be any of the usual operations?

C. Let S equal the set of positive rational numbers (i.e. numbers greater than zero). What properties does (S,x) have?

Properties of Integers in Modular Arithmetic

We will concentrate here only on integers for mod 5. The set may be denoted as:

$Z_5 = \{0,1,2,3,4\}$.

The modular operations are $\oplus$, $\otimes$, $\ominus$, and $\oslash$, corresponding to +, x, -, and ÷, as follows:

$a \oplus b = c$, if a + b divided by 5 has remainder c.

$a \otimes b = c$, if a x b divided by 5 has remainder c.

$a \ominus b = c$, if c is the unique element in Z_5 such that $a = c \oplus b$.

$a \oslash b = c$, if c is the unique element in Z_5, such that $a = c \otimes b$.

Problems

A. If a and b are in Z_5, then calculate, if possible, the following:

$a \oplus b$, $a \otimes b$, $a \ominus b$, and $a \oslash b$.

B. If c is in Z_5, calculate the $\oplus$-inverse of c and the $\ominus$-inverse of c.

C. Demonstrate which properties hold, do not hold, and do not apply to Z_5 with the modular operations.

D. Make a table for each of the following systems:

$(Z_5, \oplus)$ $(Z_5, \otimes)$ $(Z_5, \ominus)$ $(Z_5, \oslash)$

E. Let -c denote the $\oplus$-inverse of c, and let c^{-1} denote the $\otimes$-inverse of c. For each member c of Z_5 find (if possible) both -c and c^{-1}.

F. Decide which properties hold, do not hold, and do not apply for the systems listed above.

G. For which modular systems with two operations does the distributive property hold?

Properties of Subsets with Union and Intersection

The term **subset** expresses the relationship between sets. For example, let A and B be sets. A is a subset of B, if every member of A is a member of B (although every member of B may not necessarily be a member of A). The **union** of A and B is the set composed of all the members of A or B, or both A and B. Union is denoted $A \cup B$ (pronounced "A cup B"). The **intersection** of A and B is the set containing all members that belong to both A and B. Intersection is denoted $A \cap B$ ("A cap B").

Assume that a set of numbers is represented by V. In other words, $V = \{1\ 2\ 3\ 4\}$. Let *Pow* denote the subsets of V. (*Pow* stands for the power set of V.) Thus, the family members of *Pow* could be denoted:

$\varnothing = \{\ \}$
$\{1\}, \{2\}, \{3\}, \{4\}$
$\{1\ 2\}, \{1\ 3\}, \{1\ 4\}, \{2\ 3\}, \{2\ 4\}, \{3\ 4\}$
$\{1\ 2\ 3\}, \{1\ 2\ 4\}, \{1\ 3\ 4\}, \{2\ 3\ 4\}$
$\{1\ 2\ 3\ 4\}$

Pow with $\cup$ and $\cap$ is a non-number mathematical system, and each element of *Pow* can be given a simple name. For example:

Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, and $D = \{4\}$.

Then let $E=\{1\ 2\}$, $F=\{1\ 3\}$, $G=\{1\ 4\}$, $H=\{2\ 3\}$, $I=\{2\ 4\}$, and $J=\{3\ 4\}$.
Also let $K=\{1\ 2\ 3\}$, $L=\{1\ 2\ 4\}$, $M=\{1,3,4\}$ and $N=\{2,3,4\}$.
Therefore, F∪H = K; F∩H = C; and I∩F = O.

OBJECTIVES: To show what properties hold, do not hold, and do not apply to the systems (Pow,∪), (Pow,∩) and (Pow,∪,∩); to know the identity element for each operation; and to know which operation is distributed over the other.

Problems

A. Make a table for (*Pow*,∪) and (*Pow*,∩).

B. If S is in *Pow*, then let S' be all the elements in V but not in S. We call S' the complement of S (e.g. $G'=H$, $V'=0$, and $A'=N$). See how S' is related to the (∪)-inverse of S and the (∩)-inverse of S.

C. Make an (∩)-inverse table and a (∪)-inverse table.

D. In the statement of properties, when can ∪ be exchanged for ∩, and ∩ exchanged for ∪?

Properties of Counting Numbers with Operations GCF and LCM

For this mathematical system, the set of objects is N, the set of counting numbers. If a and b are in N, then let $a*b$ = GCF(a,b) and $a\#b$ = LCM(a,b).
Consider the following: 15*18 = 3 and 6#10 = 30. Here, 3 is the greatest common factor of 15 and 18, and 30 is the least common multiple of 6 and 10. Thus, (N,*) and (N,#) are mathematical systems.

OBJECTIVE: To demonstrate which properties hold, do not hold, and do not apply to the systems (N,), (N,#), and (N,*,#).*

Problem

A. Check all properties for the systems (N,*), (N,#), and (N,*,#).

Probability and Statistics

Probability theory is a rational approach for extracting reasonable information about uncertain events. Statistics is the application of probability methods to numerical data.

Sample Spaces and Probability

DEFINITION 1: In an experiment, the set of all possible outcomes is called a **sample space**.
For example, if a penny is tossed twice, the sample space, consisting of heads and tails, would be:

$$\{H,H\},\ \{H,T\},\ \{T,H\},\text{ and }\{T,T\}$$

Here, {H,T} is not the same as {T,H}. Also, if two pennies in a box are tossed on a table, the sample space would be:

$$\{H,H\},\ \{H,T\},\text{ and }\{T,T\}$$

DEFINITION 2: A particular group of outcomes from an experiment is called an **event**. A **probability** is a number associated with the events. It is defined for each event (A) as:

$$P(A) = \frac{\#\text{ of favorable cases}}{\text{total \# of distinct events}}$$

If a penny is tossed twice, what is the probability of getting a head on the second throw

(Event A)? Looking at our sample space, we see that the *favorable cases* are $\{H,H\}$ and $\{T,H\}$ and that the total number of distinct events is four. Therefore:

$$P(A) = \frac{2}{4} = \frac{1}{2}$$

If we toss a coin, it is equally likely that it will land on its head or on its tail. In the case of a die, there are six equally likely outcomes. We also know that if one outcome occurs, the others cannot occur, and therefore they are mutually exclusive.

OBJECTIVES: To determine the sample space of possible outcomes for a given random experiment and count using sample spaces, and to determine the probability of a given event in finite experiments, and compute odds and expectations.

Problems

A. Use **venn diagrams** to show the following:

1. P (A or B) = P (A) + P (B),
where A and B are mutually exclusive.

2. P (A or B) = P (A) + P (B) - P (A and B),
in all cases.

3. P (A and B) = P (A) x P (B given A)

B. Use **tree diagrams** to construct sample spaces and compute probabilities.

C. Define and study expectations for finite sample spaces.

D. Determine how many ways six books can be arranged on a shelf.

E. If you put the names of the days of the week in a box, and drew one name at random, what would be the sample space for this experiment?

List the event, consisting of all outcomes, when the day drawn will start with the letter T.

What is the probability of drawing a day that starts with T?

Measures of Central Tendency and Dispersion

DEFINITION 1: The **mean** (or arithmetic mean) of a set of data is the sum of the values divided by the total number of data points (denoted by μ). The **mode** is the number that appears most frequently in the set. The **median** is the number that represents the mid-point, when the data are arranged in a natural order, such as from the smallest number to the greatest.

Each of these three is a type of **average**. When we look at information that is presented to support particular views or conclusions, we should understand what *type* of average is being used, so that we can decide for ourselves if the average distorts the information.

Given the data set {6, 7, 8, 9, 10, 12, 15, 20, 28}, the mean is 13, the mode is 15, and the median is 11. The mode is the only value that is actually *a number* in our data set. In general, the mean and the median are rarely numbers in the set, while the mode always is. Sometimes, there is no mode at all!

DEFINITION 2: The **range** and the **standard deviation** are single numbers that tell us about the dispersion or **variance** in our data set. These numbers help us determine how well the mean, mode, or media describe the set.

The range is the difference between the largest and the smallest number in the set.

The standard deviation, however, is defined by the formula:

$$\sigma = \sqrt{\sum_{i=1}^{n} (x_i - \bar{\mu})^2 \div n}$$

where μ is the mean of $\{x_1, x_2 \ldots x_n\}$

The variance is defined as:

$$\sigma^2 = \sum_{i=1}^{n} (x_i - \bar{\mu})^2 \div n$$

(Discuss also: Binomial distribution, the correlation coefficient, and the normal curve and distribution.)

OBJECTIVES: To be able to define and compute the following: a) the mean, mode, and median for a set of data, and b) the standard deviation for a set of data.

Problems

A. Find the mean, median, mode, variance, range, and standard deviation for each of the following groups of data:

1. [10, 50, 30, 40, 10, 60, 10]
2. [5, 8, 6, 3, 5, 4, 3, 6, 1, 9]

B. If 1,000 students took a standardized test, and their scores approximated a normal curve with a mean of 600 and a standard deviation of .75, how many scored between 600 and 750?

Plane Coordinate Geometry

Plane geometry is the formal and intuitive study of figures on a plane.

We can study geometry formally by performing mathematical computations and by making logical deductions. We can also develop an intuitive understanding of the relationships between various geometric objects.

By reviewing formal mathematical skills with geometric applications and by encouraging intuitive responses to those relationships, we can better understand how the formal and intuitive approaches complement each other.

For this purpose, we will consider points and figures in the *xy*-plane, areas of lattice polygons, line segments, classifying quadrilaterals, and transformations.

Points and Figures in the xy-Plane

DEFINITION 1: When an ordered pair of numbers is given, the first number refers to the *x*-coordinate and the second to the *y*-coordinate.

In Figure 1, note the location of *A* and *B*, if *A* = (-8,5) and *B* = (-3,1). On the *x*-axis, values to the left of the *y*-axis are minus and to the right of it are plus. On the *y*-axis, values above the *x*-axis are plus and below it are minus.

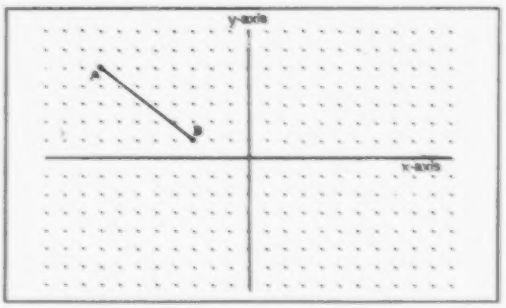

Figure 1: Plotting Coordinates.

DEFINITION 2: Given two points in the *xy*-plane (a,b) and (c,d), then the distance between the two points may be expressed as:

$$\sqrt{(c-a)^2 + (d-b)^2}$$

DEFINITION 3: If *A* and B are points in the *xy*-plane, then *AB* = *BA* denotes the line segment with endpoints *A* and *B*. Similarly, *ABCD* is a polygon, consisting of line segments *AB*, *BC*, and *CD*.

OBJECTIVES: 1. Given the coordinates of two points, expressed as an ordered pair of numbers, find: a) the locations of the points on the xy-plane and b) the distance between the points; and 2. Given the notation for a plane figure, draw it on a coordinate system.

Problems

A. Use graph paper to recreate the coordinates shown in Figure 1, then plot the following points: (1,3) (2,-4) (-7,-2) (0,-4) (-3,1) (1,-3) and (2,0).

B. Find the distance between the following pairs of points: (-3,1) (-3,-2); (4,3) (6,3); (1,3) (2,-4); (-7,-2) (-3,1); and (0,4) (2,0).

C. Let $A = (1,5)$, $B = (2,-4)$, $C = (-7,-2)$, and $D = (-3,1)$. Draw the following figures, each on its own coordinate system: *AB*, *ABCD*, *ACBD*, *ABDA*.

Areas of Lattice Polygons

DEFINITIONS: A **lattice point** is a point in the *xy*-plane with integer coordinates. A **lattice polygon** is a simple, closed polygon with all of its vertices at lattice points. A **simple closed polygon** is a polygon that forms a loop without having its sides intersect.

Assume that i is the number of points in the interior of a lattice polygon, and b is the number of points on the boundary of that polygon, the area may be expressed in the formula:

$$\text{area} = i + b/2 - 1$$

In Figure 2, if $i = 12$, and $b = 8$, the area of the polygon = 15.

OBJECTIVES: To apply the area formula to any lattice polygon, and to understand why the formula is true.

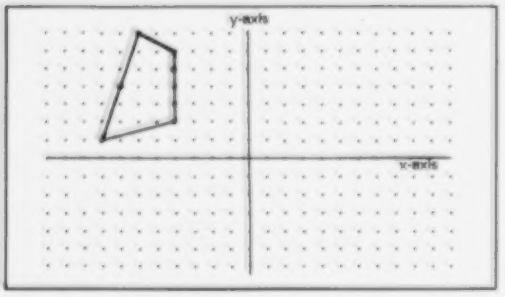

Figure 2: Areas of Polygons

Problems

A. For each of the following lattice polygons, draw it on graph paper and find its area:

1. (2,2) (6,4) (7,1) (9,6) (2,6) (2,2);

2. (1,2) (2,6) (6,5) (5,1) (1,2);

3. (1,1) (2,1) (2,3) (3,2) (4,3) (5,5)... (4,5) (3,4) (2,5) (1,5) (1,1);

4. (1,0) (7,1) (6,4) (8,4) (5,5) (5,4)... (1,5) (0,1) (3,2) (1,0)

B. Approximate π using lattice polygons.

C. Find the areas of some rectangles, using the formula. Try to see why the formula works.

Next, see why the formula works for two rectangles that are put together.

Finally, do the same for triangles.

Line Segments

DEFINITION 1: A line segment is **vertical** if its *x*-coordinates are equal, and it is **horizontal** if its *y*-coordinates are equal.

For example, the line segment (5,3) (1,3) is horizontal, and the line segment (6,0 (6,8) is vertical. Likewise, assume that $P = (a,b)$ and $Q = (c,d)$ as points in the *xy*-plane. The seg-

ment PQ is vertical if $a = c$, and it is horizontal if $b = d$.

DEFINITION 2: The **midpoint** of a line segment can be identified by its x and y coordinates. The x-coordinate of the midpoint is the **mean of the x-coordinates of the endpoints**, and the y-coordinate of the midpoint is the **mean of the y-coordinates of the endpoints**.

For example, the midpoint of (-2,6) (8,2) is (3,4). Likewise, the midpoint of PQ is the point $(a+c)/2$, $(b+d)/2$).

DEFINITION 3: The **slope** is the quotient of the difference between the y-coordinates and the x-coordinates.

For example, the slope of (-2,6) (8,2) is (6-2)/(-2-8) = 2/5. Likewise, the slope PQ is the number (a-c)/(b-d) = (c-a)/(d-b).

Vertical line segments have no slopes. Why? The slope of a horizontal line segment is also zero. Why?)

Two line segments are parallel if they have the same slope. Also, all vertical line segments are parallel.

Two line segments are perpendicular if the product of their slopes is -1. Also, all vertical line segments are perpendicular to all horizontal line segments. Geometrically, two line segments are perpendicular if the two lines determiend by the line segments intersect at right angles. For example, (2,2) (5,5) is parallel to (0,-1) (1,0), and (2,2) (5,5) is perpendicular to (0,1) (1,0).

OBJECTIVES: To determine if a line segment is vertical or horizontal, to find its midpoint and slope, and to determine if two given line segments are parallel or perpendicular.

Problems

A. Determine if the following line segments are vertical, horizontal, or neither:

(5,-2) (4,-2); (-1,0) (0,-1); and (3,3) (3,7)

B. Find the midpoint and slope of the following line segments:

(-2,5) (6,-2); (3,3) (3,7); (5,-3)(5,4); and (4,4) (-4,-4).

C. For each of the following pairs of line segments, determine if they are parallel, perpendicular, or neither:

1. (-2,-2) (0,-1) and (1,2) (5,4);

2. (-4,6 (0,-6) and (-3,5) (0,7); and

3. (-4,-1) (2,1) and (-3,2) (-2,-2).

Classification of Quadrilaterals

DEFINITIONS: A quadrilateral is a closed plane figure with four sides.

It is a **square** if its sides are equal and all its angles are right angles. It is a **rectangle** if all its angles are right angles. It is a **rhombus** if all its sides are equal. It is a **parallelogram** if its opposite sides are parallel to each other. It is a **trapezoid** if it has two sides that are parallel.

OBJECTIVES: To know the definitions for each class of quadrilateral and to be able to prove that a given quadrilateral belongs to one of the classes.

Problem

Draw quadrilaterals using the vertices below, decide to which class it belongs, and prove formally that it belongs to that class:

1. (1,2) (-1,-3) (4,-5) (6,0)
2. (7,1) (4,7) (1,1) (4,-5)
3. (8,7) (7,10) (3,2) (4,-1)
4. (7,-1) (6,3) (-2,1) (-1,-3)

Transformations

DEFINITIONS: A transformation moves a plane figure to a new location. Some of the ways in which the figure can assume new positions are by rotation, reflection, a horizontal shift, or a vertical shift.

Rotation is denoted as RO. If a figure (a,b) is rotated 90 degrees about (0,0), also known as the *origin*, it becomes $(b,-a)$. It would be written $RO(a,b) = (b,-a)$. Similarly, in **reflection** across the x-axis, $XR(a,b) = (a,-b)$, and in reflection across the y-axis, $YR(a,b) = (-a,b)$. In a **horizontal shift**, if a figure is moved by five units, then $HS5(a,b) = (a+5,b)$. Making a **vertical shift** of 3 units, $VS3(a,b) = (a,b+3)$.

To apply a transformation to a plane figure, apply the transformation to each of its vertices. For example, if A, B, and C are points, then XR applied to ABC is denoted by $SR(ABC)$ and is equal to $XR(A)\ XR(B)\ XR(C)$.

Therefore, if $A=(4,1)$, $B=(5,3)$, and $C=(6,2)$, then $XR(ABC) = (4,-1)\ (5,-3)\ (6,-2)$.

OBJECTIVES: To determine, both formally and intuitively, what happens to a figure during each type of transformation, and to discover which sequences of transformations are equivalent to other transformations.

Problems

A. Let $A=(2,0)$, $B=(3,1)$, and $C=(3,-2)$. Apply each of the transformations to ABC, and see what happens.

Given any plane figure, can you predict what will happen when one of the transformations is applied?

B. Find the transformations that must be applied to (0,0) (1,1) (1,-2) in order to get:

1. (-1,1) (0,1) (0,2);
2. (2,0) (1,1) (1,-2);
3. (1,5) (0,4) (3,4).

C. Let $A=(2,0)$, $B=(3,1)$, and $C=(3,-2)$. Convince yourself that $RO(RO(A)) = HS\text{-}4(A)$ and $VS3(RO(HS\text{-}3(ABC))) = RO(ABC)$.

4. Consider two plane figures. For each figure, find two sequences of transformations that have the same effect on the figures.

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Dr. Chachere and Dr. Gill conducted simultaneous classes for two weeks during the 1986 "MATH Alive!" summer seminar, and they followed the preceding course outline.

Teaching the Language of Mathematics

Mu'minah M. Saleem

Mathematics is a way of thinking about numbers. It has its own "language" which enables us to describe the processes we use to perform calculations and express relationships between numbers.

When students rush to solve problems in mathematics, without clearly understanding the linkage between language and process, they are likely to make mistakes and get incorrect answers. Sometimes they also say, "I know how I got it." Yet, they cannot accurately explain *how* they got the answer or *why* it is correct.

Many of the errors children make occur when numerals are given incorrect place values, even when the correct mechanics are used to add, subtract, multiply, and divide. To help them avoid these pitfalls, teachers should stress the importance of place value. In addition, teachers can develop creative word problems for manipulatives to reinforce these concepts.

Face, Place and Total Value

One of the basic concepts in which children should be drilled is the relationship between a numeral's face value, place value, and total value. The rule can be recited as:

Face times place equals total value.

For example, in the number 45$\underline{9}$, moving from right to left, the face of the underlined digit is nine, the place is the ones' place, and the total value is nine. The face of the next digit is five, it is in the tens' place, and the total value is fifty. The third digit has four as its face value, it is in the one hundreds' place, and the total value is four hundred.

The discipline of recognizing the face, place, and total value of a digit extends to all mathematical operations and helps eliminate the possibility for error. Some of these operations include determining numbers that are greater than another or performing expanded notation, finding the sum of the products, using expanded notation, regrouping in subtraction, rounding to the nearest whole number, multiplication, quotients, and decimals.

GREATER THAN. The process of learning about place value begins in kindergarten, where children can be taught that the number 12 may be stated as one ten and two ones. It is this language that helps establish the order in writing numerals. Children later will have no difficulty identifying whether one number is greater than another number. They will know that 35 is not greater than 53 because three tens are less than five tens.

EXPANDED NOTATION. Writing in expanded notation, they will find that (60 + (4 + 700)) is not 674. It is six tens, plus four ones, plus seven hundreds, or 764. And the value of the underlined digit in the number 6$\underline{3}$7 is not three but thirty. Teachers should remind students to read the numerals from the greatest to the least numerical value, regardless of the order in which they are grouped, because (4 + 700) + 60 is still 764.

SUM OF THE PRODUCTS. Finding the sum of the products can become a manageable task. Children using the correct language can see that (5x1000) + (2x100) + (8x10) + (4x1) is in fact the number 5284.

Teachers can reinforce the concept of place value by saying, "Five in the one-thousands' place, two in the one-hundreds' place, eight in the tens' place, and four in the ones' place." NOTE: It is important to refer to the "one thousands" place, not just "the thousands," because there is also a place for the "ten thousands" and another for the "one hundred thousands."

EXPONENTIAL NOTATION. Similarly, with exponential notation, the discipline of the language is again significant. Using the number 10 as the base number, show that 10 times 10 equals 100. Then 100 equals two factors of 10, which can be expressed as 10^2. This notation is called "10, exponent 2" or "the second power of 10," not "10 to the second power" as children are frequently taught in school. Also, 10^3 equals (10x10x10) or 1,000, and so on.

Therefore, when a child is given the notation, (3x104) + (8x103) + (4x102) + (3x101) + (9x100), he or she can readily determine that the number is 38,439.

REGROUPING. Another area of difficulty is regrouping in subtraction. Consider the following problem:

$$\begin{array}{r} 85 \\ -\ 7 \\ \hline 78 \end{array}$$

Children are usually taught to "borrow one from the eight" and "add it to the five" in order to subtract seven from 15. This is incorrect, and the fault lies in careless use of the language of mathematics.

It is more precise to say, "Take one ten from the eight tens, which leaves seven tens. Change the one ten back to 10 ones. Add the 10 ones to the five ones to get 15 ones. Now subtract seven ones from 15 ones."

Now consider the following problem:

$$\begin{array}{r} 8{,}000 \\ -\ \ 7 \\ \hline 7{,}993 \end{array} \quad \text{becomes} \quad \begin{array}{r} 7\ \ 9\ 9\ 10 \\ 8{,}\not{0}\ \not{0}\ \not{0}\ 0 \\ -\ \ \ \ \ 7 \\ \hline 7{,}9\,9\,3 \end{array}$$

The child should say, "I cannot subtract seven ones from zero ones, so I must go over to the eight thousand." (It is not sufficient to say "to the thousands.")

Then continue: "I take 1,000 and change it back to 10 hundreds. I add 10 hundreds to my zero hundreds, and now I have 10 hundreds.

"Since I still do not have enough ones, I take one hundred, and that leaves me with nine hundreds. The one hundred I took I must change back to 10 tens. I add 10 tens to my zero tens, and now I have 10 tens.

"I still do not have enough ones, so I take one ten from the 10 tens, and that leaves me with nine tens. The one ten I took I must change back to 10 ones. I add 10 ones to my zero ones, and now I have 10 ones. Now I can subtract the seven ones, leaving three ones.

"Finally, subtract zero tens from nine tens, leaving nine tens. Subtract zero hundreds from nine hundred, leaving nine hundred. Subtract zero thousands from seven thousand, leaving seven thousand. The answer is seven thousand, nine hundred, and ninety-three."

ROUNDING. Teachers can take the mystery out of rounding to the nearest whole number by insisting on correct language usage. For example, if the problem is to round the number 395 to the nearest tens, ask the following question: "Is 395 closer to 300 or 400?" Show the number 395 and stress the following approach:

"The least number of tens is nine; the greatest number of tens is 39; the number that tells 39 tens what to do is the five in the ones' place; and the five in the ones' place says, 'Take 39 tens up to 40 tens.' Now, forty tens equal 400. Therefore, 395 is closer to 400 than to 300."

Similarly, rounding 1428 to the nearest hundred requires the following logic. Ask the question, "Is 1428 closer to 1400 or 1500?" Show the number 1428 and stress the same approach as above:

"The least number of hundreds is four; the greatest number of hundreds is 14; the number that tells 14 hundred what to do is the two in the tens' place; and the two in the tens' place says, 'Leave the 14 hundred unchanged.' The number 1428, therefore, is closer to 1400 than to 1500."

MULTIPLICATION. Place value is critical also in multiplication. Consider the following problem:

63	becomes	$^{1}63$
x 4		x 4
252		252

A child who says, "Four ones times three ones equals 12 ones" is being precise. Twelve ones can be regrouped as one ten and two ones. Read and write the higher order for one ten, then say and write, "Two ones."

Then continue, "Four ones times six tens equals 24 tens, and 24 tens plus the one ten from the previous multiplication equals 25 tens." Twenty-five tens can also be expressed as, "Two hundreds and five tens." Write and say the higher order first, stressing the place value. Finish by saying, "Therefore, the correct answer is two hundred and fifty-two."

QUOTIENTS. Naming quotients is another subject that is facilitated by maintaining the place value of numerals. For example, to divide 3,756 by 12, say, "How many times is 12 contained in 37 hundred?" The answer is 300 times. Express the product of 300 x 12 (the quotient times the divisor) in the full place value of the dividend (the 37 hundreds being divided), as follows:

313	instead of:	313
12)3756		12)3756
3600		36
156		15
120		12
36		36
36		36
0		0

Say, "Multiplying 12 times three hundred equals 36 hundred. Subtracting 36 hundred from 37 hundred, five tens and six ones leaves 15 tens and six ones."

Then, "Multiplying 12 times one ten equals 12 tens. Subtracting 12 tens from 15 tens and six ones leaves 36 ones.

Finally, "Multiplying 12 times three ones equals 36 ones, and subtracting 36 ones from 36 ones leaves zero."

DECIMALS. Confusion can occur with decimals, too. Many people forget that the only function of the decimal point is *to locate the one's place* in the whole number.

One place (or "the first place") to the left of the ones' place is called the tens' place; one place to the right of the ones' place is the tenths' place.

The digit that is two places to the left of the ones' place is in the one hundreds' place; two places to the right is the one hundredths' place.

Three places to the left of the ones is named the one thousands' place, while three places to the right of the ones is the one thousandths' place.

Four places to the left of the ones' place is the ten thousands' place; four places to the right of the ones' place is the ten thousandths' place, and so on.

How would a teacher explain why .25 stands for twenty-five hundredths? The answer requires the use of place values:

First, note that the total value of the numeral 2 is two tenths, and the total value of the numeral 5 is five hundredths.

Also remember that the tenths must be renamed as hundredths so that like denominators can be added together. Thus:

$$\frac{2}{10} \times \frac{10}{10} = \text{twenty hundredths}$$

Then add the twenty hundredths to the five hundredths to get twenty-five hundredths:

$$\frac{20}{100} + \frac{5}{100} = \frac{25}{100}$$

Word Problems for Manipulatives

Teachers can develop their own manipulatives, using word problems that are fun for younger children. If these problems are written on flash cards, a child can turn the paper over and discover the correct answer. The manipulatives then become self-checking tests, and child can get a bonus for each one answered correctly. Some of these word problems are:

I tell you to add. I don't make a fuss. I look like a t. *I'm a sign you can trust. Who am I?* (A plus sign)

I'm a stick with two round eyes. Use me for ten ten tries. Who am I? (The number 100)

I look like a V. *I point to the right. The number to the left has all the might. Who am I?* (The "greater than" sign)

More advanced students might prefer problems like these:

I am thinking of a pair of numbers whose sum is 10, whose difference is 6, whose product is 16, and whose quotient is 4. What are they? (Eight and two)

I am thinking of a pair of numbers, the sum of which is 16, the difference is zero, the product is 64, and the quotient is 1. What are they? (Eight and eight)

Providing children with the tools they need gives them confidence in the accuracy of their work. This is the first step in motivating them to like mathematics and to work quickly and accurately when taking tests.

These positive feelings then can be reinforced by incentives, such as rubber stamps with words like "Fantastic" or "Right On, Superstar" which the teacher can put on worksheets or assessment activities when children demonstrate acceptable performance.

Children should also see that the teacher likes to take tests because they are a challenge and fun. In fact, a good motto for teachers is:

We take the tests given to students,
and we lead by performance.

The point in stressing place value and creating word problem manipulatives is this. It is important that children learn how to reason, rather than get an answer by merely guessing. Equally as important, however, children should learn how to explain the processes they use to someone else, for the greatest knowledge acquired is that which we can share with others.

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Her special concerns include the relationships between number pairs, as well as the laws, properties, and language of mathematics.

Early African Mathematics for the Classroom

Walter M. Young

One way to inspire African-American youth to learn mathematics is to introduce them to the many accomplishments Africans have made to mathematics. We can show them that African influences may be seen today in our numeration system, calculation techniques, network theory, mathematical games, number patterns, games of chance, and architecture.

Many Greek philosophers and mathematicians studied in Africa, where Black Africans were an integral part of all highest levels of civilization, from the workers to the planners and the pharoahs.

Thales visited Egypt in the 6th Century B.C. and studied with the Egyptian priests before he introduced geometry into Greece. Pythagoras, on the advice of his teacher, Thales, also visited Egypt and Babylon and was initiated by priests there into their "mysteries." The learning they acquired on the banks of the Nile was incorporated by them and others into Western thought, enabling the Greeks to begin a new era of intellectual energy and activity.

Eurocentric mathematicians frequently limit the study of mathematics to the study of abstract structures, mathematics as science, and mathematics *for the sake of mathematics*. However, there is evidence that this approach has led to the denial of African contributions and to pedagogical techniques that may be alienating many young people from studying mathematics and from pursuing mathematics-related careers.

The African approach to mathematics, however, is rooted in practical problems of society and in religious rituals. By using historical works as background material, students can learn the methods of counting and enjoy mathematical games that were created by Africans long ago. Following are some of the ancient mathematical experiences that you can use to challenge young people in your classrooms.

Egyptian Numerals in Hieroglyphics

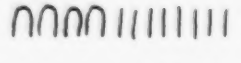
When you introduce your students to Egyptian hieroglyphic symbols and have them express Hindu-Arabic numerals as Egyptian numerals and *vice versa*, they will be able to appreciate the economy of the Hindu-Arabic numeration system and understand the contributions made to it by the Egyptian numeration system. The material that follows is intended to be used in conjunction with the book *Africa Counts: Number and Pattern in African Culture*, by Claudia Zaslavsky (Lawrence Hill & Company, 1973) and similar reference books.

Learning the Symbols

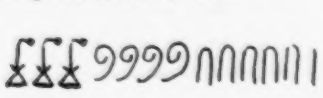
Some of the symbols used in Egyptian hieroglyphics are shown below:

1		1,000	
10		10,000	
100		100,000	

To represent 48 as an Egyptian numeral, you would write:



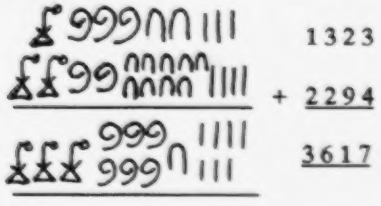
To represent 3,451, write:



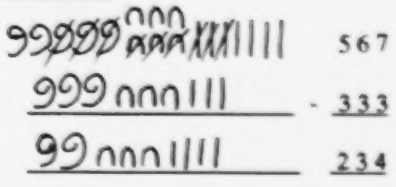
Activities

Perform the operations indicated below, but and note that the complexity of the problems you demonstrate will depend on the grade level of the student:

Addition:

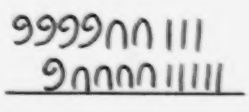


Subtraction:

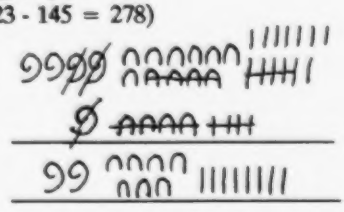


Regrouping in Subtraction

In the Original:
(423 minus 145)



Regrouped:
(423 - 145 = 278)



Calculating by Duplation and Mediation

Multiplication and division were based on the principles of **duplation** (doubling) or **mediation** (halving), as well as the fact that all numbers may be expressed as a sum of the powers of two.

You can present the algorithms, or step-by-step procedures to your students by starting with single-digit multiplication and division. Then work with double digits and extend the exercise to include additional powers of two. Later, ask your students to perform all calculations using Egyptian numerals.

You should point out also that the process of duplation and mediation today is utilized by high-speed electronic calculating machines.

Duplation

Duplation can be used for both multiplication and division.

Multiplication

To multiply 13 by 24, using duplation, make a table with two columns, one column headed by the number 1 and the other by 24.

Double the value in each column and place an asterisk beside each number in the first column required to make 13. Find the sum of the corresponding numbers in the second column, as follows:

*1	24
2	48
*4	96
*8	192
16	384

Since 1 + 4 + 8 = 13, the sum of the corresponding numbers (24 + 96 + 192) = 312.

Division

To divide 972 by 36, using duplation, use successive subtraction. First, make two columns, as shown previously:

1	36
2	72
4	144
8	288
16	576
32	1152

The find the number in column two that is closest to but smaller than 972, and then subtract it from 972. Continue taking each new total and performing successive subtractions until the final difference is zero. Identify the corresponding values in column one, and find their sum, as follows:

Subtraction	*Sum of the Corresponding Values*
972-576 = 396	16
396-288 = 108	8
108- 72 = 36	2
36- 36 = 0	1
Therefore, the quotient is	27

Mediation

Mediation also can be used for both multiplication and division.

Multiplication

To multiply 13 by 24, using mediation, also make a table with two columns, but this time one is headed by 13 and the other by 24.

Halve the successive values in the column headed by 13, disregarding remainders, and double the successive values in the column headed by 24.

Find the sum of each number in column two associated with an odd number in column one.

*13	24
6	48
* 3	96
* 1	192

Thus, 13 x 24 = 24 + 96 + 192 = 312

Division

The process of division is similar to multiplication. Again, make a table with two columns, one headed by 1 and the other by the divisor. Double successive values in both columns until the column headed by the divisor is greater than or equal to the dividend.

Identify the values in the second column which, when added together, equal the value of the dividend. Then find the sum of the corresponding values in the first column.

The Yoruba System of Numeration

The ancient Yoruba people, from the area now known as southern Nigeria, developed a way to add and multiply from a base of 20. Each number had a distinct name, which was

expressed as a word in the Yoruba language. They did not use symbols like those used in Egyptian hieroglyphics.

They used the numbers 1 to 10, as well as 20, 30, 200, and 400. All other numbers were derived as compounds or multiples of these named numbers.

For example, 11 = 10 + 1.
Likewise, 45 = (20 x 3) - 10 - 5

Even Multiples of 10

Numbers that are even multiples of 10, up to 200, were formed by expressing them as multiples of 20:

20 = A distinct term	120 = 20 x 6
40 = 20 x 2	140 = 20 x 7
60 = 20 x 3	160 = 20 x 8
80 = 20 x 4	180 = 20 x 9
100 = 20 x 5	200 = A distinct term

Odd Multiples of 10

Numbers that are odd multiples of 20 were formed by subtracting 10 from the even multiples of 20, except in the case of 30, which had its own distinct name.

30 = A distinct term	130 = (20 x 7) - 10
50 = (20 x 3) - 10	150 = (20 x 8) - 10
70 = (20 x 4) - 10	170 = (20 x 9) - 10
90 = (20 x 5) - 10	190 = 200 - 10
110 = (20 x 6) - 10	200 = A distinct term

Multiples of 200

Numbers from 200 up to 2000 were formed as multiples of 200, except for 400 (20 x 20), which is known as the square of all the digits. It had the distinct name of *irinwo* or *erinwo*, the "elephant of figures" or the highest coined word in calculation. However, the system becomes irregular after the number 200.

Activities

Explain the logical structure of the Yoruba system of numeration, and ask your students to express numerals between 1 and 200, then up to one million.

African Mathematical Games

Africans have always used games and activities from daily life to strengthen their knowledge of numerals. Children learned about numbers and mathematics by using beans, pebbles, stones and other objects, and various activities such as buying and selling in the marketplace. Two of the most popular games are Kpelle and Mancala.

Kpelle: The Game of Arrangement

The Kpelle children of Liberia play a game involving sixteen stones in two rows of eight each. One person (the player) is sent away, and the others choose a stone.

When the player returns, he must determine the stone that was selected by the others. To do so, the player asks questions that require a yes or no answer, such as:

"Is the stone in Row 1 or Row 2?"

After the reply, the player rearranges the stones in the two rows. This sequence may be repeated up to three more times, and the player must identify the stone by the fourth reply.

Strategy. The player's strategy in this game, after the first reply, is to rearrange eight stones

so that half the original 16 stones are placed in different rows.

Players can start by exchanging odd numbered stones, but they can also exchange stones with other number relationships, such as even numbers, prime numbers, composite numbers, multiples of numbers, and so on.

When the player asks the question for the second time, if the stone has not moved, it must be one of the original eight. The player then interchanges four stones (*four of the original eight, if the stone has not moved; otherwise, four new ones*).

After the third reply, the player similarly interchanges two stones to narrow the options even further. The answer given the fourth time the question is asked determines exactly which stone has been chosen.

Playing this game enables students to apply the principles of doubling and halving, identify number relationships, and understand the importance of memory training.

Mancala: The Game of Transferring

Mancala is played in many African countries. It has different names, and rules vary from place to place. However, the game is most frequently played with a board having two rows of six holes and an additional hole at each end. (*You may substitute egg cartons in your classroom.*) Four objects, such as beans or stones, are placed in each of the 12 holes.

One player takes all four objects from any hole on his side of the board and, moving in a counter-clockwise direction, drops one in each of the next four holes. If the last object falls in a hole already containing objects, and if the number of objects satisfies a predetermined condition (e.g. being an odd number, a prime number, a number divisible by three, etc.) , the player picks up all the objects from the hole.

The player repeats the process as long as the condition is met. If the condition is not met, the next player takes a turn. The game is over when a player captures a predetermined number of objects, such as a simple majority or any other number, such as 25 objects.

Playing this game teaches one-to-one correspondence in the counting process, number relationships, numbers that satisfy more than one relationship, and order relationships.

Weights and Measures

Societies with extensive commercial activity tend to have standardized measurements. Songhai, for example, in the late 15th century, achieved a high level of uniformity in standardizing weights and measures. They also relied on a network of inspectors to enforce the standards.

For the most part, however, Africans relied on units of length that referred to the human body. Some of these measurements include:

Inch: The distance from the tip of the thumb to the knuckle.

Cubit: The distance from the elbow to the end of the middle finger (or six palms).

Palm: The distance from the outer edge of the little finger to the outer edge of the index finger, stretching across the back of the hand.

Span: The distance between the thumb and the little finger, when the hand is outstretched to the point of slight tension.

Foot: Four palms or sixteen fingers.

Yard: One pace. (Take a normal size step and measure the distance from the toes of one foot to the toes of the other foot.

Fathom: The distance from the tip of the fingers on the left hand to the tip of the fingers

on the right hand, when both arms are outstretched at the sides of the body.

The human body also has been used to measure some units of capacity. In ancient Ethiopia, for example, they used a **handful** and an **armful**. On the other hand, some measurements depended on the article itself.

Activities

Present the units of measurement based on the human body, and have each student use a ruler to measure his or her own body in inches.

Create a wall chart, listing the children in the left column and the dimensions for selected units of measurement in columns across the chart. Then ask if there is a *standard* for the class.

Use the chart to teach how a mean value for each part of the body in the chart can be calculated, and this can be used to arrive at some *reasonable* basis for standardization within the group.

Older children may develop a frequency distribution for the measured inches, then compute the mean, median and mode values for each frequency distribution. They may also be asked to enter the data on a computer and plot a graph to illustrate what they first derived from manual computations.

Determine a reasonable number of beans that could be used for standardizing "a handful of beans."

Discuss with the class the problems that can occur without standardization of weights and measurements. Inform them that many African countries adopted the British Imperial standards in the 19th century, and show some of the ways in which those standards were different from the metric standards and from the standards used in the U.S.A.

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